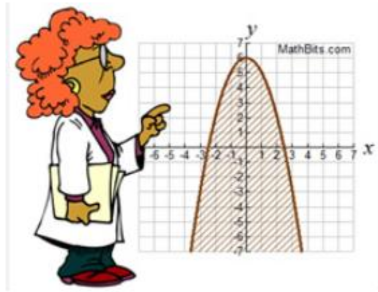
QUADRATIC INEQUALITIES



For linear inequalities such as $7n + 4 \geq 7$, the Basic Principle of Inequalities (If you multiply or divide by a negative number, switch the inequality sign) is sufficient. But other problems will require a technique called the Boundary Point Method. This will help us solve inequalities such as $x^2 - 3x - 10 < 0$.

❑ *THE BOUNDARY POINT METHOD*

1) Change the *inequality* to an *equation* and solve the equation.

2) The solutions of that equation are called the *boundary points*.

3) Plot the boundary points on the real line. This will create one or more *intervals* on the line.

4) Now choose a *test point* from each interval (any convenient number in the interval) and plug it into the original inequality. If the test point satisfies the inequality, then the entire interval in which the test point lies is a part of the solution of the inequality. If the test point does not satisfy the inequality, then the associated interval is not part of the solution.

5) Lastly, check the boundary points themselves in the inequality.

❑ *SOLVING QUADRATIC INEQUALITIES*

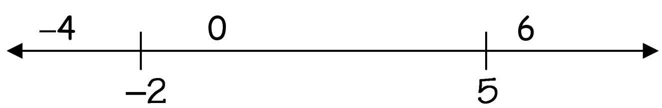
<u>EXAMPLE 1:</u> **Solve for x: $x^2 - 3x - 10 \geq 0$**

<u>Solution:</u> We solve this kind of problem using the Boundary Point Method. First we find the boundary points:

$$x^2 - 3x - 10 = 0 \Rightarrow (x+2)(x-5) = 0 \Rightarrow x = -2 \text{ or } x = 5.$$

Now we mark these boundary points on a number line, and then choose a test point in each interval to test in the original inequality $x^2 - 3x - 10 \geq 0$:

The **boundary** points are –2 and 5.



The **test** points are –4, 0, and 6.

$\mathbf{-4} \in (-\infty, -2)$ $\quad\quad$ $\mathbf{0} \in (-2, 5)$ $\quad\quad$ $\mathbf{6} \in (5, \infty)$

$(-4)^2 - 3(-4) - 10 = 16 + 12 - 10 = 18$, which is ≥ 0. Thus, $(-\infty, -2)$ <u>is</u> part of the solution.

$0^2 - 3(0) - 10 = 0 - 0 - 10 = -10$, which is not ≥ 0. So, $(-2, 5)$ is <u>not</u> part of the solution.

$6^2 - 3(6) - 10 = 36 - 18 - 10 = 8$, which is ≥ 0. Therefore, $(5, \infty)$ <u>is</u> part of the solution.

Last, you can check the boundary points yourself — they work in the original inequality. Our final answer is the combination of the left interval and the right interval:

$$\boxed{(-\infty, -2] \cup [5, \infty)}$$

The answer can also be written

$$\{x \in \mathbb{R} \mid x \leq -2 \text{ OR } x \geq 5\}$$

EXAMPLE 2: **Solve for w: $9w - w^2 \geq 14$**

Solution: Solve the associated equation: $9w - w^2 = 14$. Rearranging the terms and multiplying through by -1 gives the equation $w^2 - 9w + 14 = 0$, whose solutions are 2 and 7.

These two boundary points yield the three intervals to check: $(-\infty, 2)$, $(2, 7)$ and $(7, \infty)$. Choose a test point (your choice) in each interval and substitute that test point into the original inequality. You'll find that the only interval that works is the center one: $(2, 7)$.

Finally, check the boundary points themselves. They both work in the original inequality. The final answer, then, is $\{ w \in \mathbb{R} \mid 2 \leq w \leq 7 \}$, which we write as

$$\boxed{[2, 7]}$$

EXAMPLE 3: **Solve for y: $y^2 - 12 > -4y$**

Solution: To solve this inequality, convert it to an equation, write it in standard quadratic form, factor, and you'll find that the boundary points are -6 and 2. Pick some test points and see that the intervals $(-\infty, -6)$ and $(2, \infty)$ work, but the interval $(-6, 2)$ doesn't. Also, the boundary points themselves don't work. The final answer is therefore

$$\boxed{(-\infty, -6) \cup (2, \infty)}$$

EXAMPLE 4: **Solve for x: $x^2 + x + 4 > 0$**

Solution: The associated equation is $x^2 + x + 4 = 0$, which won't factor, so let's use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(4)}}{2(1)} = \frac{-1 \pm \sqrt{-15}}{2}$$

There are no real solutions to this equation; therefore, there are no boundary points, and so there is only one interval to check, namely the entire real line, $(-\infty, \infty)$.

Choose any real number [that is, any number in the interval $(-\infty, \infty)$], for instance, **2**, as a test point, and plug it into the original inequality: $2^2 + 2 + 4 = 10$, which is clearly > 0. Since the test point worked, the interval in which the test point lies must work; however, this interval was the entire real line. Therefore, the solution to the inequality is the set of real numbers

$$\boxed{\mathbb{R}}$$

EXAMPLE 5: **Solve for x: $x^2 + 6x + 9 < 0$**

Solution: Solving the associated equation, $x^2 + 6x + 9 = 0$, we obtain one solution: $x = -3$. That produces two intervals: $(-\infty, -3)$ and $(-3, \infty)$. Pick a test point in each interval and you'll discover that both test points fail; even the boundary point $x = -3$ fails. In other words, everything fails! The inequality has **NO solution** at all. We can write this using the null set (empty set):

$$\boxed{\varnothing}$$

EXAMPLE 6: **Solve for x: $x^2 + 6x + 9 \le 0$**

Solution: Solving the associated equation gives us the same boundary point as in the previous example. The two intervals also fail, as in the previous example. BUT, this time, if we check the boundary point, $x = -3$, it works. So our final solution of the inequality is

$\boxed{x = -3}$ [which can also be written $\{-3\}$.]

Homework

Solve each quadratic inequality:

1. $a^2 - 9a - 10 > 0$
2. $2x^2 \le 5 - 9x$
3. $x^2 + 9 < 6x$
4. $y^2 + 10y + 25 \le 0$
5. $z^2 > 12z - 36$
6. $w^2 + 14w + 49 \ge 0$
7. $x^2 \ge 9$
8. $2x^2 - 15x + 7 < 0$
9. $5a - a^2 \ge 6$
10. $-3(3n^2 + 8n) \le 16$

Review Problems

Solve each inequality:

11. $2(x - 5) - 9(2x + 1) > 0$

12. a. $x^2 + 7x + 12 < 0$ b. $x^2 - 100 \geq 0$
c. $x^2 + 10x + 25 > 0$ d. $x^2 - 6x + 9 \leq 0$
e. $4x^2 - 12x + 9 < 0$ f. $x^2 \geq 2x - 1$

13. Hint: 14. $x^2 - 5x \geq -4$

15. $x^2 < 20x - 100$

16. True/False:

a. When solving a linear inequality, one needs to reverse the inequality symbol only when dividing each side by a negative number.

b. The solution of the inequality $ax \leq b$ is $x \leq \frac{b}{a}$.

c. The solution of $x^2 \geq 3x + 10$ is $[-2, 5]$.

d. The solution of $n^2 + 10n + 25 \geq 0$ is $\mathbb{R}$.

Solutions

1. $(-\infty, -1) \cup (10, \infty)$ **2.** $\left[-5, \frac{1}{2}\right]$ **3.** $\varnothing$

4. $\{-5\}$ **5.** $(-\infty, 6) \cup (6, \infty)$; or $\mathbb{R} - \{6\}$

6. $\mathbb{R}$ **7.** $(-\infty, -3] \cup [3, \infty)$

8. $\left(\frac{1}{2}, 7\right)$ **9.** $[2, 3]$ **10.** $\mathbb{R}$

11. $\left(-\infty, -\frac{19}{16}\right)$

12. a. $(-4, -3)$ b. $(-\infty, -10] \cup [10, \infty)$ c. $\mathbb{R} - \{-5\}$

d. $\{3\}$ e. $\varnothing$ f. $\mathbb{R}$

13. Not solvable — it depends on whether a positive or negative.

14. $(-\infty, 1] \cup [4, \infty)$ **15**. $\varnothing$

16. a. F b. F c. F d. T

"A child without education is like a bird without wings."

—Tibetan Proverb